

## Inhomogeneous linear differential equations

Suppose we are given a **linear differential equation** (LDE) of order  $N$  or, more generally, the following linear system of first order differential equations

$$\dot{\mathbf{x}} + \mathbf{A}\mathbf{x} = \mathbf{b} \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (1)$$

where  $\mathbf{x} = (x_1, x_2, \dots, x_N)^t$  is the column vector of unknown functions of  $t$ ,  $\mathbf{A}$  is a  $N \times N$  matrix (possibly  $t$  dependent),  $\mathbf{b} = (b_1, b_2, \dots, b_N)^t$  is a vector of known functions and  $\mathbf{x}_0$  is an initial condition.

Suppose we have solved the **homogenous problem**

$$\dot{\mathbf{x}} + \mathbf{A}\mathbf{x} = \mathbf{0}$$

and found the  $N$  (linearly independent) solution vectors guaranteed by Cauchy's initial-value theorem,  $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_N$

$$\dot{\mathbf{y}}_i + \mathbf{A}\mathbf{y}_i = \mathbf{0} \quad i = 1, 2, \dots, N$$

that we arrange as columns of a matrix  $\mathbf{Y} = \mathbf{Y}(t)$  in such a way that

$$\dot{\mathbf{Y}} + \mathbf{A}\mathbf{Y} = \mathbf{0}$$

holds.

We are now in a position to find the particular solution of the **non-homogeneous problem** above. To this end, consider the following change of variables

$$\mathbf{x} = \mathbf{Y}\mathbf{z}$$

in the non-homogeneous problem. Since it holds

$$\mathbf{b} = (\dot{\mathbf{Y}} + \mathbf{A}\mathbf{Y})\mathbf{z} + \mathbf{Y}\dot{\mathbf{z}} \equiv \mathbf{Y}\dot{\mathbf{z}}$$

the DE is readily solved for  $\mathbf{z}$  to give

$$\mathbf{z}(t) = \mathbf{z}(t_0) + \int_{t_0}^t \mathbf{Y}^{-1}(t')\mathbf{b}(t')dt'$$

Here  $\mathbf{Y}(t)$  is non-singular (for any  $t$ ) since it is made up of linearly independent column vectors. Finally, we find for  $\mathbf{x}$

$$\mathbf{x}(t) = \mathbf{Y}(t)\mathbf{z}(t_0) + \int_{t_0}^t \mathbf{Y}(t)\mathbf{Y}^{-1}(t')\mathbf{b}(t')dt'$$

where

$$\mathbf{x}(t_0) = \mathbf{Y}(t_0)\mathbf{z}(t_0) \equiv \mathbf{x}_0$$

hence

$$\mathbf{x}(t) = \mathbf{Y}(t)\mathbf{Y}^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^t \mathbf{Y}(t)\mathbf{Y}^{-1}(t')\mathbf{b}(t')dt'$$

We have thus found the solution in the form

$$\boxed{\mathbf{x}(t) = \mathbf{\Pi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^t \mathbf{\Pi}(t, t')\mathbf{b}(t')dt'} \quad (2)$$

where

$$\mathbf{\Pi}(t, t') = \mathbf{Y}(t)\mathbf{Y}^{-1}(t') \quad (3)$$

is the **fundamental propagator** of the LDE. This function propagates a give state at time  $t'$  to a time  $t$ , that is  $\mathbf{x}(t) = \mathbf{\Pi}(t, t')\mathbf{x}'$  is the solution of the homogeneous LDE that reduces to  $\mathbf{x}'$  when  $t \rightarrow t'$ .

The general solution of Eq. 2 is given at any time  $t$  as superposition of a 'freely' evolving state

$$\mathbf{x}^0(t) \equiv \mathbf{\Pi}(t, t_0)\mathbf{x}_0$$

and a 'forced' one

$$\bar{\mathbf{x}}(t, t') = \mathbf{\Pi}(t, t')\mathbf{b}(t')$$

namely,

$$\mathbf{x}(t) = \mathbf{x}^0(t) + \int_{t_0}^t \bar{\mathbf{x}}(t, t')dt'$$

This latter term admits the following simple interpretation. Let us consider for definiteness  $t \geq t_0$  and the following 'driving force'

$$\mathbf{b}(t) = \mathbf{B}\delta(t - \bar{t})$$

where  $\bar{t} \in (t_0, t)$ . Clearly, under this condition the LDE is homogeneous for *both*  $t < \bar{t}$  and  $t > \bar{t}$  and thus its solution is of the general form  $\mathbf{x}(t) = \mathbf{\Pi}(t, t')\mathbf{x}(t')$ . For  $t < \bar{t}$  the initial condition is just  $\mathbf{x}_0$  and thus  $\mathbf{x}(t) \equiv \mathbf{x}^0(t)$  (as defined above) holds up to a value of  $t$  which is infinitesimally smaller than  $\bar{t}$ , where the solution takes the value  $\bar{\mathbf{x}}^-$ . For  $t = \bar{t}$  the solution undergoes to a sudden jump to  $\bar{\mathbf{x}}^+$  due to the 'kick'  $\mathbf{b}(t)$  and then for  $t > \bar{t}$  it propagates freely according to  $\mathbf{x}(t) = \mathbf{\Pi}(t, \bar{t})\bar{\mathbf{x}}^+$ . The state right *after* the kick,  $\bar{\mathbf{x}}^+$ , can be obtained by the integral form of the LDE

$$\mathbf{x}(t) = \bar{\mathbf{x}}^- - \int_{\bar{t}}^t \mathbf{A}(t')\mathbf{x}(t')dt' + \int_{\bar{t}}^t \delta(t' - \bar{t})\mathbf{B}dt'$$

by taking the limit  $t \rightarrow \bar{t}^+$  and reads

$$\bar{\mathbf{x}}^+ = \bar{\mathbf{x}}^- + \mathbf{B}$$

Hence for  $t > \bar{t}$

$$\mathbf{x}(t) = \mathbf{\Pi}(t, \bar{t})\bar{\mathbf{x}}^- + \mathbf{\Pi}(t, \bar{t})\mathbf{B} \equiv \mathbf{\Pi}(t, t_0)\mathbf{x}_0 + \bar{\mathbf{x}}(t, \bar{t})$$

as discussed above.

Formally, for several (physical) reasons it may be convenient to separate the **causal** ( $t > t_0$ ) from the **non-causal** ( $t < t_0$ ) evolution (this is helpful when  $t$

is a sort of time). This may be accomplished by defining the **retarded** ( $\mathbf{G}^R$ ) and the **advanced** ( $\mathbf{G}^A$ ) **Green's functions**

$$\mathbf{G}^R(t, t') = \Theta(t - t')\mathbf{\Pi}(t, t') \quad \mathbf{G}^A(t, t') = \Theta(t - t')\mathbf{\Pi}(t, t')$$

where  $\Theta(x)$  is the Heaviside function,  $\Theta(x) = 1$  for  $x \geq 0$  and  $\Theta(x) = 0$  otherwise. These functions can be used to re-write the general solution above as

$$\mathbf{x}(t) = \mathbf{G}^R(t, t_0)\mathbf{x}_0 + \int_{t_0}^{+\infty} \mathbf{G}^R(t, t')\mathbf{b}(t')dt' \quad \text{for } t \geq t_0 \quad (4)$$

and

$$\mathbf{x}(t) = \mathbf{G}^A(t, t_0)\mathbf{x}_0 + \int_{t_0}^{-\infty} \mathbf{G}^A(t, t')\mathbf{b}(t')dt' \quad \text{for } t \leq t_0 \quad (5)$$

where the upper limit of the integration has been change to  $+\infty$  ( $-\infty$ ) since  $\Theta(t-t')$  ( $\Theta(t'-t)$ ) guarantees that the integrand vanishes for  $t' > t$  ( $t' < t$ ). The Green's functions satisfy simple LDEs that can be obtained by direct derivation. For instance,

$$\frac{\partial \mathbf{G}^R(t, t')}{\partial t} = \delta(t - t')\mathbf{\Pi}(t, t') + \Theta(t - t')\frac{\partial \mathbf{\Pi}(t, t')}{\partial t} \equiv \delta(t - t')\mathbf{1} - \mathbf{A}\mathbf{G}^R(t, t')$$

i.e.,

$$\frac{\partial \mathbf{G}^R(t, t')}{\partial t} + \mathbf{A}\mathbf{G}^R(t, t') = \delta(t - t')\mathbf{1} \quad (6)$$

where we have used  $\mathbf{\Pi}(t', t') = \mathbf{1}$  and the fundamental property of the propagator

$$\frac{\partial \mathbf{\Pi}(t, t')}{\partial t} + \mathbf{A}\mathbf{\Pi}(t, t') = \mathbf{0}$$

Notice that  $\mathbf{G}^R$  is discontinuous since it undergoes a sudden jump for equal values of its arguments

$$\mathbf{G}^R(t, t') \equiv \mathbf{0} \quad \text{for } t < t' \quad \text{and} \quad \lim_{t \rightarrow t'^+} \mathbf{G}^R(t, t') = \mathbf{1}$$

Importantly, *defining*  $\mathbf{G}^R$  by means of Eq. 6 it is easy to verify that<sup>1</sup>

$$\mathbf{x}(t) = \mathbf{G}^R(t, t_0^-)\mathbf{x}_0 + \int_{t_0}^{+\infty} \mathbf{G}^R(t, t')\mathbf{b}(t')dt'$$

is a solution of the LDE of Eq. 1 for any  $t \geq t_0$ , and similarly for  $\mathbf{G}^A$  and  $t \leq t_0$ . Indeed, upon defining  $\mathbf{D}_t = \frac{d}{dt} + \mathbf{A}$ , we have

$$\mathbf{D}_t \mathbf{G}^R(t, t_0^-) \equiv \mathbf{0} \quad \text{for } t \geq t_0$$

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<sup>1</sup> $t_0^-$  is here introduced to properly handle the limit  $t \rightarrow t_0$ .

and

$$D_t \int_{t_0}^{+\infty} \mathbf{G}^R(t, t') \mathbf{b}(t') dt' = \int_{t_0}^{+\infty} D_t \mathbf{G}^R(t, t') \mathbf{b}(t') dt' = \int_{t_0}^{+\infty} \delta(t-t') \mathbf{b}(t') dt' = \mathbf{b}(t)$$

where  $D_t$  could be moved within the integral thanks to the causality condition (differently from Eq. 2). On the other hand, it also holds

$$\lim_{t \rightarrow t_0} \mathbf{G}^R(t, t_0^-) \mathbf{x}_0 = \mathbf{x}_0$$

thereby proving that the above solution satisfies the desired initial condition.

## Homogeneous linear differential equation

In view of the above, unless we are brave enough to directly seek the Green's function of the problem, it is clear that the simplest strategy to solve the LDE is first to find the solutions of the homogeneous differential equation

$$\dot{\mathbf{x}} + \mathbf{A}\mathbf{x} = \mathbf{0} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

and then build up the fundamental propagator, Eq. 3. There is no general strategy for this, unless  $\mathbf{A}$  is constant, *i.e.* it does not depend on  $t$ . We start considering this case, and the rather common situation in which  $\mathbf{A}$  is *diagonalizable*<sup>2</sup>. Under such circumstances, any vector  $\mathbf{x}$  can be written as a superposition of the eigenvectors  $\mathbf{u}_k$  of the matrix  $\mathbf{A}$  and thus we can solve separately for each of them and later combine the eigen-solutions. Let us suppose then  $\mathbf{x}_0 = \mathbf{u}_k$  and seek for a solution of the form

$$\mathbf{x}_k(t) = c_k(t) \mathbf{u}_k$$

The equation reads as

$$(\dot{c}_k(t) + \alpha_k c_k(t)) \mathbf{u}_k = \mathbf{0}$$

where  $\alpha_k$  is the corresponding eigenvalue, and its solution follows simply as

$$c_k(t) = \exp(-\alpha_k(t - t_0))$$

In general, upon noticing that for arbitray  $\mathbf{x}_0$  it holds

$$\mathbf{x}_0 = \sum_k \mathbf{u}_k \langle \tilde{\mathbf{u}}_k, \mathbf{x}_0 \rangle$$

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<sup>2</sup>Remember that  $\mathbf{A}$  is said diagonalizable if there exist a non-singular matrix  $\mathbf{X}$  such that  $\mathbf{X}^{-1} \mathbf{A} \mathbf{X} = \mathbf{A}_D$  where  $\mathbf{A}_D$  is a diagonal matrix. Since this amounts to  $\mathbf{A} \mathbf{X} = \mathbf{X} \mathbf{A}_D$ ,  $\mathbf{X}$  is the matrix of the eigenvectors of  $\mathbf{A}$  (the columns of  $\mathbf{X}$ ) and these are guaranteed to be linearly independent (since  $\mathbf{X}$  is non-singular). In other words, the eigenvectors of a diagonalizable matrix  $\mathbf{A}$  form a basis of the vector space on which  $\mathbf{A}$  operates. Non-diagonalizable matrices are 'rare' — they have zero measure in a certain sense — and need to be handled case by case. A *necessary* condition for  $\mathbf{A}$  to be non-diagonalizable is the existence of degenerate eigenvectors, *i.e.* of eigenspaces of dimension  $> 1$ .

where  $\{\tilde{\mathbf{u}}_k\}$  is the dual basis of  $\{\mathbf{u}_k\}$ , the solution for an arbitrary  $\mathbf{x}_0$  is readily found to be

$$\mathbf{x}(t) = \sum_k e^{-\alpha_k(t-t_0)} \mathbf{u}_k \langle \tilde{\mathbf{u}}_k, \mathbf{x}_0 \rangle$$

The set of  $N$  linearly independent solutions can thus be obtained by choosing  $N$  linearly independent initial states, *e.g.*, the set of canonical vectors  $\mathbf{x}_0 = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\}$  or, simpler, the set of eigenvectors  $\mathbf{u}_k$ . In the latter case we directly obtain

$$\mathbf{Y}(t) = \mathbf{X} \text{diag} \{e^{-\alpha_1 t}, e^{-\alpha_2 t}, \dots, e^{-\alpha_N t}\}$$

where  $\mathbf{X}$  is matrix of the (column) eigenvectors, and

$$\mathbf{\Pi}(t, t') = \mathbf{X} \text{diag} \{e^{-\alpha_1(t-t')}, e^{-\alpha_2(t-t')}, \dots, e^{-\alpha_N(t-t')}\} \mathbf{X}^{-1}$$

Equivalently,

$$\boxed{\mathbf{\Pi}(t, t') = \exp(-\mathbf{A}(t - t'))} \quad (7)$$

where use has been made of the definition of a function of a (diagonalizable) matrix

$$f(\mathbf{A}) = \mathbf{X} f(\mathbf{A}_D) \mathbf{X}^{-1}$$

the function of a diagonal matrix being defined simply as

$$f(\mathbf{A}_D) = \text{diag} \{f(\alpha_1), f(\alpha_2), \dots, f(\alpha_N)\}$$

This expression can also be seen as the result of the direct integration of the equation

$$\frac{\partial \mathbf{\Pi}(t, t_0)}{\partial t} + \mathbf{A} \mathbf{\Pi}(t, t_0) = \mathbf{0} \quad \mathbf{\Pi}(t, t_0) = \mathbf{1}$$

provided the exp function is defined as above.

When  $\mathbf{A}$  is  $t$ -dependent the method sketched above cannot be applied, and no simple exp function can be defined that solves the LDE. To proceed, we re-write the equation in integral form

$$\mathbf{\Pi}(t, t_0) = \mathbf{1} - \int_{t_0}^t \mathbf{A}(t') \mathbf{\Pi}(t', t_0) dt'$$

and solve it by iteration

$$\begin{aligned} \mathbf{\Pi}(t, t_0) = & \mathbf{1} - \int_{t_0}^t dt_1 \mathbf{A}(t_1) + \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \mathbf{A}(t_1) \mathbf{A}(t_2) + \\ & ..(-1)^n \int_{t \geq t_1 \geq \dots \geq t_n \geq t_0} dt_1 .. dt_n \mathbf{A}(t_1) .. \mathbf{A}(t_n) + .. \end{aligned}$$

Equivalently, we can write

$$\boxed{\mathbf{\Pi}(t, t_0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{[t_0, t]^n} dt_1 .. dt_n \mathbf{T}(\mathbf{A}(t_1) .. \mathbf{A}(t_n))}$$

where  $\mathsf{T}$  is a  $t$ -ordering operator defined as

$$\mathsf{T}(\mathbf{A}(t_1)..\mathbf{A}(t_n)) = \mathbf{A}(t_{s_1})..\mathbf{A}(t_{s_n})$$

$(t_{s_1}, t_{s_2}, ..t_{s_n})$  being the ordered permutation of the original  $t$  values,  $t_{s_1} \geq t_{s_2} \geq ..t_{s_n}$ . This expression represents a formal solution for the fundamental propagator of the LDE. Though rarely useful in practice it represents a very useful starting point for investigating the LDE and devising approximations. Clearly, it reduces to Eq. 7 when  $\mathbf{A}$  is  $t$ -independent.

Before concluding, it is worth commenting on a common strategy to solve a homogeneous, constant-coefficient linear equation of order  $N$ , that is with the help of the characteristic polynomial, since this method is closely related to (though slightly less general than) the above 'spectral' method. To show this let us consider the following LDE

$$x^{(N)} + a_1 x^{(N-1)} + ..a_{N-1} x^{(1)} + a_N x^{(0)} = 0$$

where  $x^{(k)}$  denotes the  $k^{\text{th}}$  derivative of  $x$ . The characteristic equation is obtained by seeking solutions of the form  $x(t) \propto \exp(\beta t)$  and reads as

$$\beta^N + a_1 \beta^{N-1} + ..a_{N-1} \beta + a_N = q_N(\beta) = 0$$

where  $q_N(\beta) = \sum_k^N a_{N-k} \beta^k$  is the characteristic polynomial of the homogeneous differential equation. The solutions of this algebraic equation provide the exponents that define the solutions of the LDE. They are  $N$  in number if the zeros of  $q_N(\beta)$  are simple, otherwise there remain a number of linearly independent solutions that need to be found by other means. To connect this settings with our previous findings we make the standard replacement

$$\mathbf{x} = (x^{(0)}, x^{(1)}, ..x^{(N-1)})$$

and rewrite the LDE in the standard form

$$\dot{\mathbf{x}} + \mathbf{A}\mathbf{x} = 0$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & & & \\ & 0 & -1 & & \\ & & 0 & -1 & .. \\ & & .. & .. & -1 \\ a_N & a_{N-1} & .. & a_2 & a_1 \end{bmatrix}$$

The relevant eigenvalue problem reads as

$$p_N(\alpha) = \det(\alpha \mathbf{1} - \mathbf{A}) = \det \begin{bmatrix} \alpha & +1 & & & \\ & \alpha & +1 & & \\ & & \alpha & +1 & .. \\ & & .. & .. & +1 \\ -a_N & -a_{N-1} & .. & -a_2 & \alpha - a_1 \end{bmatrix}$$

where the determinant on the right hand side can be expanded as

$$p_N(\alpha) = \alpha p_{N-1}(\alpha) + (-1)^{N+1}(-a_N) \det \begin{bmatrix} +1 & 0 & & & \\ \alpha & +1 & 0 & & \\ & \alpha & +1 & 0 & \dots \\ & & \dots & \dots & 0 \\ & & & \alpha & +1 \end{bmatrix} \equiv \alpha p_{N-1}(\alpha) + (-1)^N a_N$$

Here the polynomial  $p_{N-1}(\alpha)$  reads similar to  $p_N(\alpha)$  as

$$p_{N-1}(\alpha) = \det \begin{bmatrix} & \alpha & & +1 & & \\ & & \alpha & +1 & & \\ & & & \alpha & +1 & \dots \\ & & & \dots & \dots & +1 \\ -a_{N-1} & -a_{N-2} & \dots & -a_2 & \alpha - a_1 \end{bmatrix}$$

and can be expanded similarly to above to give

$$\begin{aligned} p_N(\alpha) &= \alpha (\alpha p_{N-1}(\alpha) + (-1)^{N-1} a_{N-1}) + (-1)^N a_N = \dots = \\ &= \sum_{k=0}^N (-1)^{N-k} a_{N-k} \alpha^k \quad (\text{with } \alpha_0 := 1) \end{aligned}$$

Thus,  $q_N(\beta) \equiv (-1)^N p_N(-\beta)$  and the eigenvalues  $\mathbf{A}$  are just the opposite of the roots of the characteristic polynomial  $q_N$  (consistently with the minus sign in Eq. 7).

With the same token, it is not hard to show that  $\mathbf{x} = (1, \beta, \beta^2, \dots, \beta^{N-1})^t$  is eigenvector of  $\mathbf{A}$  with eigenvalue  $-\beta$  when  $\beta$  is a root of the characteristic polynomial

$$\beta^N + a_1 \beta^{N-1} + \dots + a_{N-1} \beta + a_N \equiv q_N(\beta) = 0$$

since in that case  $a_1 \beta^{N-1} + \dots + a_{N-1} \beta + a_N \equiv -\beta^N$  solves the non-trivial condition in the  $N^{\text{th}}$  row of the eigenvalue problem, *i.e.*,

$$\begin{bmatrix} 0 & -1 & & & \\ & 0 & -1 & & \\ & & 0 & -1 & \dots \\ & & \dots & \dots & -1 \\ a_N & a_{N-1} & \dots & a_2 & a_1 \end{bmatrix} \begin{bmatrix} 1 \\ \beta \\ \beta^2 \\ \dots \\ \beta^{N-1} \end{bmatrix} = \begin{bmatrix} -\beta \\ -\beta^2 \\ -\beta^3 \\ \dots \\ -\beta^N \end{bmatrix} \equiv -\beta \begin{bmatrix} 1 \\ \beta \\ \beta^2 \\ \dots \\ \beta^{N-1} \end{bmatrix}$$